ON ERDÉLYI-MAGNUS-NEVAI CONJECTURE FOR JACOBI POLYNOMIALS

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ABSTRACT. T. Erdélyi, A.P. Magnus and P. Nevai conjectured that for $\alpha, \beta \geq -\frac{1}{2}$, the orthonormal Jacobi polynomials $\mathbf{P}_k^{(\alpha,\beta)}(x)$ satisfy the inequality

$$\max_{x\in[-1,1]} (1-x)^{\alpha+\frac{1}{2}}(1+x)^{\beta+\frac{1}{2}} \left(\mathbf{P}_k^{(\alpha,\beta)}(x)\right)^2 = O\left(\max\left\{1, (\alpha^2+\beta^2)^{1/4}\right\}\right),$$

[Erdélyi et al.,Generalized Jacobi weights, Christoffel functions, and Jacobi polynomials, SIAM J. Math. Anal. 25 (1994), 602-614]. Here we will confirm this conjecture in the ultraspherical case $\alpha = \beta \geq \frac{1+\sqrt{2}}{4}$, even in a stronger form by giving very explicit upper bounds. We also show that

$$\sqrt{\delta^2 - x^2}\,(1-x^2)^{\alpha} \left(\mathbf{P}_{2k}^{(\alpha,\alpha)}(x)\right)^2 < \frac{2}{\pi}\left(1 + \frac{1}{8(2k+\alpha)^2}\right)$$

for a certain choice of δ, such that the interval $(-\delta, \delta)$ contains all the zeros of $\mathbf{P}_{2k}^{(\alpha,\alpha)}(x)$. Slightly weaker bounds are given for polynomials of odd degree.

Keywords: Jacobi polynomials

1. INTRODUCTION

In this paper we will use bold letters for orthonormal polynomials versus regular characters for orthogonal polynomials in the standard normalization [14].

Given a family $\{\mathbf{p}_i(x)\}$ of orthonormal polynomials orthogonal on a finite or infinite interval I with respect to a weight function $w(x) \geq 0$, it is an important and difficult problem to estimate $\sup_{x\in I} \sqrt{w(x)}\,|\mathbf{p}_i(x)|$, or, more generally, to find an envelope of the function $\sqrt{w(x)}\,\mathbf{p}_i(x)$ on I. Those two questions become almost identical if we introduce an auxiliary function $\phi(x)$ such that $\sqrt{\phi(x)w(x)}\,\mathbf{p}_i(x)$ exhibits nearly equioscillatory behaviour. Of course, the existence of such a function is far from being obvious but it turns out that in many cases one can choose $\phi = \sqrt{(x-d_m)(d_M - x)}$, with d_m, d_M being appropriate approximations to the least and the largest zero of p_i respectively. The simplest example is given by Chebyshev polynomials $T_i(x)$ and $\phi = \sqrt{1-x^2}$. This illustrates a classical result of G. Szegö asserting that for a vast class of weights on $[-1,1]$ and $i \to \infty$, the function $\sqrt{\sqrt{1-x^2}\,w(x)}\,\mathbf{p}_i(x)$ equioscillates between $\pm\sqrt{\frac{2}{\pi}}$, [14].

A very general theory for exponential weights $w = e^{-Q(x)}$ stating that under some technical conditions on Q,

$$\max_I \left| \sqrt{\sqrt{|(x-a_{-i})(a_i - x)|}\,w(x)}\,\mathbf{p}_i(x) \right| < C,$$

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where the constant C is independent on i and $a_{\pm i}$ are Mhaskar-Rahmanov-Saff numbers for Q, was developed by A.L. Levin and D.S. Lubinsky [11]. Recently it has been extended to the Laguerre-type exponential weights $x^{2\rho}e^{-2Q(x)}$ [6, 12].

In the case of classical orthogonal Hermite and Laguerre polynomials explicit bounds confirming such a nearly equioscillatory behaviour independently on the parameters involved were given in [8] and [9] respectively.

The case of Jacobi polynomials $P_k^{(\alpha,\beta)}(x), \quad w(x) = (1-x)^\alpha(1+x)^\beta$, is much more difficult. Let us introduce some necessary notation.

We define

$$M_k^{\alpha,\beta}(x, d_m, d_M) = \sqrt{(x-d_m)(d_M-x)}\, (1-x)^\alpha(1+x)^\beta \left(\mathbf{P}_k^{(\alpha,\beta)}(x)\right)^2,$$

$$\mathcal{M}_k^{\alpha,\beta}(d_m, d_M) = \max_{x\in[-1,1]} M_k^{\alpha,\beta}(x; d_m, d_M),$$

what we will abbreviate to $M_k^{\alpha,\beta}(x)$ and $\mathcal{M}_k^{\alpha,\beta}$ if $d_m = -1, \ d_M = 1$, that is for $\phi(x) = \sqrt{1-x^2}$. We will also omit one of the superscripts in the ultraspherical case $\alpha = \beta$ writing, for example, $M_k^{\alpha}(x)$ instead of $M_k^{\alpha,\alpha}(x)$, and shorten $M_k^\alpha(x,-d,d)$, $\mathcal{M}_k^\alpha(-d,d)$ to $M_k^\alpha(x,d)$, $\mathcal{M}_k^\alpha(d)$ respectively.

As $P_k^{(\alpha,\beta)}(x) = (-1)^k P_k^{(\beta,\alpha)}(-x)$ we may safely assume that $\alpha \ge \beta$.

For $-\frac{1}{2} < \beta \le \alpha < \frac{1}{2}$, the following is known [3]:

$$\mathcal{M}_k^{\alpha,\beta} \le \frac{2^{2\alpha+1}\Gamma(k+\alpha+\beta+1)\Gamma(k+\alpha+1)}{\pi k!\,(2k+\alpha+\beta+1)^{2\alpha}\Gamma(k+\beta+1)} = \frac{2}{\pi} + O\left(\frac{1}{k}\right), \tag{1}$$

where $k = 0, 1, \ldots$.

A slightly stronger inequality in the ultraspherical case was obtained earlier by L. Lorch [13].

A remarkable result covering almost all possible range of the parameters has been established by T. Erdélyi, A.P. Magnus and P. Nevai, [5],

$$\mathcal{M}_k^{\alpha,\beta} \le \frac{2e\left(2+\sqrt{\alpha^2+\beta^2}\right)}{\pi}, \tag{2}$$

provided $k \ge 0, \ \ \alpha, \beta \ge -\frac{1}{2}$.

Moreover, they suggested the following conjecture:

Conjecture 1.

$$\mathcal{M}_k^{\alpha,\beta} = O\left(\max\left\{1, |\alpha|^{1/2}\right\}\right),$$

provided $\alpha \ge \beta \ge -\frac{1}{2}$.

The best currently known bound was given by the author [7],

$$\mathcal{M}_k^{\alpha,\beta} \le 11\left(\frac{(\alpha+\beta+1)^2(2k+\alpha+\beta+1)^2}{4k(k+\alpha+\beta+1)}\right)^{1/3} = O\left(\alpha^{2/3}\left(1+\frac{\alpha}{k}\right)^{1/3}\right), \tag{3}$$

provided $k \ge 6, \ \ \alpha \ge \beta \ge \frac{1+\sqrt{2}}{4}$.

We also brought some evidences in support of the following stronger conjecture

Conjecture 2.

$$\mathcal{M}_k^{\alpha,\beta} = O\left(\max\left\{1, |\alpha|^{1/3}\left(1+\frac{|\alpha|}{k}\right)^{1/6}\right\}\right),$$

provided $\alpha \geq \beta \geq -\frac{1}{2}$.

Here we will confirm this conjecture in the ultraspherical case. Namely we prove the following

Theorem 1. *Suppose that* $k \geq 6,\ \ \alpha = \beta \geq \frac{1+\sqrt{2}}{4}$. *Then*

$$\mathcal{M}_k^{\alpha} < \mu\, \alpha^{1/3} \left(1 + \frac{\alpha}{k}\right)^{1/6}, \tag{4}$$

where

$$\mu = \begin{cases} \frac{10}{7}, & k \ \ \text{even}, \\ 22, & k \ \ \text{odd}. \end{cases}$$

We deduce this result from the following two theorems. The first, which has been established in [7], gives a sharp inequality for the interval containing all the local maxima of the function $M_k^{\alpha,\beta}(x)$. The second one will be proven here and in fact demonstrates equioscillatory behaviour of $M_k^{\alpha}(x,d)$ under an appropriate choice of d.

Theorem 2. *Suppose that* $k \geq 6,\ \ \alpha \geq \beta \geq \frac{1+\sqrt{2}}{4}$. *Let* x *be a point of a local extremum of* $M_k^{\alpha,\beta}(x)$. *Then* $x \in (\eta_{-1}, \eta_1)$, *where*

$$\eta_j = j\left(\cos(\tau + j\omega) - \theta_j \left(\frac{\sin^4(\tau + j\omega)}{2\cos\tau\cos\omega}\right)^{1/3} (2k + \alpha + \beta + 1)^{-2/3}\right) \tag{5}$$

$$\sin\tau = \frac{\alpha + \beta + 1}{2k + \alpha + \beta + 1}, \ \ \sin\omega = \frac{\alpha - \beta}{2k + \alpha + \beta + 1}, \ \ 0 \leq \tau, \omega < \frac{\pi}{2};$$

and

$$\theta_j = \begin{cases} 1/3, & j = -1, \\ 3/10, & j = 1. \end{cases}$$

In particular, in the ultraspherical case

$$|x| < \eta = \cos\tau \left(1 - \frac{2^{-1/3}}{3}\, (2k + 2\alpha + 1)^{-2/3}\, tan^{4/3}\tau\right), \tag{6}$$

with $\sin\tau = \frac{2\alpha+1}{2k+2\alpha+1}$.

Theorem 3. *Suppose that* $\alpha > \frac{1}{2}$, *and let*

$$\delta = \sqrt{1 - \frac{4\alpha^2 - 1}{(2k + 2\alpha + 1)^2 - 4}}. \tag{7}$$

Then

$$\mathcal{M}_k^{\alpha}(\delta) < \begin{cases} \frac{2}{\pi}\left(1 + \frac{1}{8(k+\alpha)^2}\right), & k \geq 2, \ \ \text{even}, \\ \frac{230}{\pi}, & k \geq 3, \ \ \text{odd}. \end{cases} \tag{8}$$

Moreover, all local maxima of the function $M_k^{\alpha}(x)$ *lie inside the interval* $(-\delta, \delta)$.

To prove this theorem we construct an envelope of $M_k^{\alpha,\beta}(x; d_m, d_M)$ using so-called Sonin's function. Then we show that in the ultraspherical case for $\alpha > \frac{1}{2}$ it has the only minimum at $x = 0$ if $\delta_m = -1,\ \delta_M = 1$, whereas for $-d_m = d_M = \delta$ the point $x = 0$ is the only maximum. Sharper bounds for the even case are due to the fact that $x = 0$ is the global maximum of $M_{2k}^{\alpha}(x, \delta)$ and the value of $P_{2k}^{(\alpha,\alpha)}(0)$ is known.

The paper is organized as follows. In the next section we present a simple lemma being our main technical tool. We will illustrate it by proving that the function $M_k^{\alpha\ \beta}(x)$ is unimodal with the only minimum in a point depending only on α and β. The even and the odd cases of Theorem 3 will be proven in sections 3 and 4 respectively. The last section deals with the proof of Theorem 1.

2. Preliminaries

In his seminal book [14] Szegö presented a few result concerning the behaviour of local extrema of classical orthogonal polynomials based on an elementary approach via so-called Sonin's function. In particular, he gave a comprehensive treatment of the Laguerre polynomials [14, Sec 7.31, 7.6], but did not try to deal with the Jacobi case for arbitrarily values of α and β. Here we combine his approach with the following very simple idea.

Given a real function $f(x)$, Sonin's function $S = S(f;x)$ is $S = f^2 + \psi(x)f'^2$, where $\psi(x) > 0$ on an interval $\mathcal{I}$ containing all local maxima of f. Thus, they lie on S, and if S is unimodal we can locate the global one.

Lemma 4. *Suppose that a function f satisfies on an open interval $\mathcal{I}$ the Laguerre inequality*

$$f'^2 - ff'' > 0, \tag{9}$$

and a differential equation

$$f'' - 2A(x)f' + B(x)f = 0, \tag{10}$$

where $A \in \mathbb{C}(\mathcal{I}),\ \ B(x) \in \mathbb{C}^1(\mathcal{I})$, and B has at most two zeros on $\mathcal{I}$. Let

$$S(f;x) = f^2 + \frac{f'^2}{B},$$

then all the local maxima of f in $\mathcal{I}$ are in the intervals defined by $B(x) > 0$, and

$$Sign\left(\frac{d}{dx}S(f;x)\right) = Sign(4AB - B').$$

Proof. We have $0 < f'^2 - ff'' = f'^2 - 2Aff' + Bf^2$, hence $B(x) > 0$ whenever $f' = 0$. Finally,

$$\frac{d}{dx}\left(f^2 + \frac{f'^2}{B}\right) = \frac{4AB - B'}{B^2}\, f'^2(x),$$

and $B(x) \neq 0$ in one or two intervals containing all the extrema of f on $\mathcal{I}$. $\square$

Let us make a few remarks concerning the Laguerre inequality (9). Usually it is stated for hyperbolic polynomials, that is real polynomials with only real zeros, and their limiting case, so-called Polya-Laguerre class. In fact, it holds for a much vaster class of functions. Let $L(f) = f'^2 - ff''$, defining $\mathcal{L} = \{f(x) : L(f) > 0\}$, we observe that $\mathcal{L}$ is closed under linear transformations $x \to ax + b$. Moreover, since

$$L(fg) = f^2L(g) + g^2L(f),$$

$\mathcal{L}$ is closed under multiplication as well. Thus, $L(x^\alpha) = \alpha x^{2\alpha-2}$, yields the polynomial case and much more. Many examples may be obtain by $L\left(e^f\right) = -e^{2f} f''$ and obvious limiting procedures.

For our purposes it is enough that (9) holds for the functions

$$((x-d_m)(d_M-x))^{1/4}\,(1-x)^{\alpha/2}(1+x)^{\beta/2}P_k^{(\alpha,\beta)}(x),$$

provided $-1 \le d_m < x < d_M \le 1$, and $\alpha, \beta \ge 0$.

To demonstrate how powerful this lemma is, we apply it to $M_k^{\alpha,\beta}(x)$ to show that its local maxima lie on a unimodal curve.

From the differential equation for Jacobi polynomials

$$(1-x^2)y'' = ((\alpha+\beta+2)x+\alpha-\beta)y' - k(k+\alpha+\beta+1)y;\;\; y = P_k^{(\alpha,\beta)}(x), \tag{11}$$

we obtain

$$\begin{gathered} 4(1-x^2)^2 z'' = 4x(1-x^2)z' - \\ \left[(2k+\alpha+\beta+1)^2(1-x^2) - 2(1+x)\alpha^2 - 2(1-x)\beta^2 + 1\right] z; \\ z = (1-x)^{\frac{\alpha}{2}+\frac{1}{4}}(1+x)^{\frac{\beta}{2}+\frac{1}{4}}y, \quad z^2 = M_k^\alpha(x). \end{gathered} \tag{12}$$

Thus, in the notation of Lemma 4,

$$A(x) = \frac{x}{2(1-x^2)}\,,$$

$$B(x) = \frac{(2k+\alpha+\beta+1)^2(1-x^2) - 2(1+x)\alpha^2 - 2(1-x)\beta^2 + 1}{4(1-x^2)^2}\,.$$

Now we calculate

$$D = 2(1-x^2)^3(4AB - B') = (\alpha^2-\beta^2)(x^2+1) + (2\alpha^2+2\beta^2-1)x. \tag{13}$$

Theorem 5. *For $\alpha \ge \beta > \frac{1}{2}$, the consecutive maxima of the function $M_k^{\alpha,\beta}(x)$ decrease for $x < x_0$ and increase for $x > x_0$, where*

$$x_0 = \frac{\sqrt{4\beta^2-1} - \sqrt{4\alpha^2-1}}{\sqrt{4\beta^2-1} + \sqrt{4\alpha^2-1}}.$$

Proof. It is enough to show that the function $S(z;x)$ is unimodal with the only minimum at x_0.

Since $B_1 = 4(1-x^2)B(x)$, the numerator of B, is a quadratic with the negative leading coefficient, by lemma 4 it suffices to verify that x_0 is the only zero of $D(x)$ in the region defined by $B_1(x) > 0$.

For, we calculate $B_1(-1) = 1 - 4\beta^2 \le 0$, $B_1(1) = 1 - 4\alpha^2 \le 0$, and

$$B_1\left(\frac{\beta-\alpha}{\alpha+\beta+1}\right) = \frac{(2\alpha+1)(2\beta+1)\left((2k+1)(2k+2\alpha+2\beta+1)+1\right)}{(\alpha+\beta+1)^2} > 0.$$

Since

$$\frac{\beta-\alpha}{\beta+\alpha+1} \in [-1,1],$$

$B(x)$ has precisely two zeros on $[-1,1]$.

It is easy to check that D has two real zeros for $\alpha, \beta > \frac{1}{2}, \alpha \ne \beta$. Moreover, for $\alpha \ne \beta$,

$$D(-1) = 1 - 4\beta^2 < 0, \;\; D(1) = 4\alpha^2 - 1 > 0,$$

hence only the largest zero of D lies between the zeros of B_1. If $\alpha = \beta$, then $D = 0$ implies $x = 0$, and

$$B_1(0) = (2k+1)(2k+2\alpha+2\beta+1)+1 > 0,$$

leading to the same conclusion. This completes the proof. $\square$

Remark 1. *Let $-1 < x_1 < \dots < x_k < 1$, be the zeros of $P_k^{(\alpha\beta)}(x)$. According to Theorem 5 the global extremum of $M_k^{\alpha,\beta}(x)$ lies in one of the intervals $[\eta_{-1}, x_1]$, $[x_k, \eta_1]$, where $\eta_{\pm 1}$ are given by (5). Rather accurate bounds χ_{-1} and χ_1 on x_1 and x_k, such that $x_1 < \chi_{-1} < \chi_1 < x_k$, and $|\eta_j - \chi_j| = O\left((k+\alpha+\beta)^{-2/3}\right)$, $j = \pm 1$, were given in* [10].

3. Proof of Theorem 3, even case

In this section we prove Theorem 3 for ultraspherical polynomials of even degree. Without loss of generality we will assume $x \geq 0$.

To simplify some expressions it will be convenient to introduce the parameter $r = 2k + 2\alpha + 1$.

The required differential equation for

$$g = (d^2 - x^2)^{1/4}(1-x^2)^{\alpha/2}, \;\; g^2 = M_k^{\alpha}(x, -d, d),$$

is

$$g'' - 2A(x)g' + B(x)g = 0,$$

where

$$A(x) = \frac{x(2d^2 - 1 - x^2)}{2(d^2-x^2)(1-x^2)},$$

$$B(x) = \frac{(1-x^2)r^2 - 4\alpha^2}{4(1-x^2)^2} + \frac{2d^2 - d^4 + (3-4d^2)x^2}{4(1-x^2)(d^2-x^2)^2}.$$

We also find

$$D(x) = \frac{2(d^2-x^2)^3(1-x^2)^2}{x}\,(4AB - B') =$$

$$\left(4\alpha^2 - (1-d^2)r^2\right)(d^2-x^2)^2 + (3-4d^2)x^4 - 2(5d^4 - 9d^2 + 3)x^2 - d^6 + 9d^4 - 9d^2.$$

In what follows we choose $d = \delta$, where δ is defined by (7). Notice that it can be also written as

$$\delta = \sqrt{\frac{r^2 - 4\alpha^2 - 3}{r^2 - 4}}.$$

The following lemma shows that δ is large enough to include all oscillations of $M_k^{\alpha}(x)$. This fact is crucial for our proof of Theorem 1.

Lemma 6. *The interval $(-\delta, \delta)$ contains all local maxima of $M_k^{\alpha}(x)$, provided $\alpha > \frac{1}{2}$.*

Proof. The assumption $\alpha > \frac{1}{2}$ implies that δ is real for $k \geq 0$. It is an immediate corollary of a general result given in [7] (eq. (17) for $\lambda = 0$), that in the ultraspherical case and $k, \alpha \geq 0$, all local maxima of $M_k^{\alpha}(x)$ lie between the zeros of the equation

$$A_0(x) = 4k(k+2\alpha+1) - \left((2k+2\alpha+1)^2 + 4\alpha + 2\right)x^2 = 0.$$

Since, as easy to check, $A_0(\delta) > 0$, the local maxima are confined to the interval $(-\delta, \delta)$. $\square$

To apply Lemma 4 we shell check the relevant properties of B and D, what will be accomplished in the following to lemmas.

Lemma 7. *Let $\alpha > \frac{1}{2}\,,\ \ k \geq 1$, then for $d = \delta$ the equation $B(x) = 0$ has the only real positive zero $x_0,\ \ \delta < x_0 < 1$. In particular, $B(x) > 0$ for $0 < x < \delta$.*

Proof. It is easy to check that $r^2 - 4\alpha^2 > 3,\ \ r^2 > 4$, for $\alpha > \frac{1}{2}\,,\ \ k \geq 1$. The numerator B_1 of $B(x)$ is

$$B_1(x) = -r^2x^6 + \left((1+2\delta^2)r^2 + 4\delta^2 - 4\alpha^2 - 3\right)x^4 -$$

$$\left((\delta^4 + 2\delta^2)r^2 - \delta^4 - 8\alpha^2\delta^2 + 6\delta^2 - 3\right)x^2 + \left(\delta^2 r^2 - 4\alpha^2\delta^2 - \delta^2 + 2\right)\delta^2.$$

Using Mathematica we find the discriminant of this polynomial in x,

$$Dis_x(B_1) = \frac{(r^2 - 4\alpha^2 - 3)\left((r^2 - 4\alpha^2 - 2)^2 + 2r^2 - 9\right)(24\alpha^2 - 6)^6\, r^8}{(r^2 - 4)^{14}}\, R^2(\alpha, r),$$

where

$$R(\alpha, r) =$$

$$100(r^2 - 4\alpha^2)^2\alpha^2 r^2 + 7r^6 - (976\alpha^2 + 90)r^4 + (5456\alpha^4 + 3180\alpha^2 + 375)r^2 - 4(12\alpha^2 + 5)^3.$$

Under our assumptions the expressions $r^2 - 4\alpha - 3$ and $(r^2 - 4\alpha^2 - 2)^2 + 2r^2 - 9$ are positive. Furthermore, rewriting $R(\alpha, r)$ in terms of k and α one can checks that the substitution $\alpha \to \alpha + \frac{1}{2}$ gives a polynomial consisting of monomials of the same sign. Thus, for any $k > 0$ and $\alpha > \frac{1}{2}$ the discriminant does not vanish and the equation $B_1(x) = 0$ has the same number of real zeros. For $\alpha = k = 1$ we obtain the following test equation with just two real zeros,

$$804 - 2733x^2 + 3150x^4 - 1225x^6 = 0.$$

It is left to demonstrate that the only positive zero x_0 of the equation $B_1(x) = 0$, is in the interval $(\delta, 1)$. For, we verify

$$B_1(\delta) = 5(1 - \delta^2)^2\delta^2 > 0, \quad B_1(1) = -4\alpha^2(1 - \delta^2)^2 < 0.$$

This completes the proof. $\square$

Lemma 8. *Let $\alpha > \frac{1}{2}\,,\ \ k \geq 1$ and $0 < x < \delta$, then $D(x) < 0$.*

Proof. We find

$$\frac{(r^2 - 4)^3}{3(4\alpha^2 - 1)}D(x) = 2(r^2 - 4)(2r^2 - 12\alpha^2 - 5)x^2 - (r^2 - 4\alpha^2 - 3)(4r^4 - 4\alpha^2 - 15).$$

Then

$$D(0) < 0,\ \ D(\delta) = -5(4\alpha^2 - 1)(r^2 - 4\alpha^2 - 3) < 0,$$

and the result follows. $\square$

Applying two previous lemmas and Lemma 4 we obtain the following result.

Lemma 9. *For $x \geq 0$ the local maxima of $M_k^\alpha(x, \delta)$ form a decreasing sequence. In particular, $\mathcal{M}_k^\alpha(\delta) = M_k^\alpha(0, \delta)$.*

Remark 2. *The value of δ has been found as a solution of the equation $Dis_x D = 0$. Surprisingly, it is split into linear and biquadratic factors. Besides trivial zeros $d = 0, 1$, this equation has four positive roots $d_1 < d_2 < d_3 < d_4$, where d_1 is of order $O\left(\frac{1}{\sqrt{k(k+\alpha)}}\right)$. The other three are very close, in fact*

$$d_3 - d_2 = O\left(\frac{1}{k^{3/2}\sqrt{k+\alpha}}\right), \quad d_4 - d_3 = O\left(\frac{\alpha^2}{k^{3/2}(k+\alpha)^{5/2}}\right).$$

We have chosen the simplest one $\delta = d_3$.

To prove the inequality

$$M_k^\alpha(\delta) < \frac{2}{\pi}\left(1 + \frac{1}{8(k+\alpha)^2}\right), \tag{14}$$

we have to find $M_k^\alpha(0,\delta)$. The value of $P_k^{(\alpha,\alpha)}(0)$ for even k is (see e.g. [1]),

$$P_k^{(\alpha,\alpha)}(0) = (-1)^{k/2} \frac{\Gamma(k+\alpha+1)}{2^k \left(\frac{k}{2}\right)! \Gamma(\frac{k}{2}+\alpha+1)}. \tag{15}$$

This yields

$$\mathbf{P}_k^{(\alpha,\alpha)}(0) = (-1)^{k/2} \frac{\sqrt{r\, k!\, \Gamma(r-k)}}{2^{r/2}\left(\frac{k}{2}\right)!\, \Gamma(\frac{r-k+1}{2})}.$$

To simplify this expression we use the following inequality (see e.g. [2]),

$$\frac{\Gamma(x+1)}{\Gamma^2(\frac{x}{2}+1)} < \frac{2^{x+\frac{1}{2}}}{\sqrt{\pi(x+\frac{1}{2})}}, \quad x \geq 0, \tag{16}$$

what yields for $k + 2\alpha \geq 0$,

$$\left(\mathbf{P}_k^{(\alpha,\alpha)}(0)\right)^2 < \frac{2r}{\pi\sqrt{(2k+1)(r+2\alpha)}}.$$

Hence, for $|x| \leq \delta$, we have

$$\mathcal{M}_k^\alpha(\delta) = M_k^\alpha(0,\delta) = \delta\left(\mathbf{P}_k^{(\alpha,\alpha)}(0)\right)^2 < \sqrt{\frac{r^2-4\alpha^2}{r^2-4}}\, \frac{2r}{\pi\sqrt{(2k+1)(r+2\alpha)}}.$$

It is an easy exercise to check that for $k \geq 2, \ \alpha \geq \frac{1}{2}$, the last expression does not exceed

$$\frac{2}{\pi}\left(1 + \frac{1}{8(k+\alpha)^2}\right).$$

This proves the even case of Theorem 3.

Remark 3. *In* [5] *the following pointwise bound on $M_k^{\alpha,\beta}(x)$ is given.*

$$M_k^{\alpha,\beta}(x) < \frac{2e}{\pi}\, \frac{(2k+2\alpha+2\beta+1)(2k+2\alpha+2\beta+2)}{(2k+2\alpha+2\beta+2)^2 - \frac{2\alpha^2}{1-x} - \frac{2\beta^2}{1+x}}. \tag{17}$$

For the ultraspherical case this yields

$$M_k^\alpha(0) < \frac{2e}{\pi}\left(1 + O\left(\frac{\alpha^2}{k(k+\alpha)}\right)\right).$$

Thus, (17) is quite precise, provided $\alpha = O(k)$.

4. Proof of Theorem 3, odd case

In this section we will establish the odd case of Theorem 3 by reducing it to the previous one. We also give slightly more accurate bounds under the assumptions $k \geq 7, \quad \alpha \geq \frac{1+\sqrt{2}}{4}$. They will be used in the proof of Theorem 1 in the next section.

As δ is a function of k and α, to avoid ambiguities or a messy notation arising when they vary, throughout this section we will use $\delta(k,\alpha)$ instead of δ and set $\mathcal{F}_k^\alpha = \mathcal{M}_k^\alpha(\delta)$, and $F_k^\alpha(x) = M_k^\alpha(x,\delta)$.

Since the value of the first, nearest to zero, maximum of $F_k^\alpha(x)$, which we assume is attained at $x = \xi$, is unknown for odd k, we need some technical preparations. First of all we have to find an upper bound on ξ. Let $k = 2i+1$ be odd, and let $0 = x_0 < x_1 < \ldots < x_i$, be the nonnegative zeros of $P_k^{(\alpha,\alpha)}(x)$. Obviously, $0 < \xi < x_1$, so we can use an upper bound on x_1 instead. An appropriate estimate for zeros of ultraspherical polynomials has been given in [4], in particular

$$x_1 < \left(\frac{2k^2+1}{4k+2} + \alpha \right)^{-1/2} h_k,$$

where h_k is the least positive zero of the Hermite polynomial $H_k(x)$.

Since $h_k \leq \sqrt{\frac{21}{4k+2}}$, [14, sec. 6.3], we obtain

$$\xi \leq \sqrt{\frac{21}{2k^2+4\alpha k+2\alpha+1}} := \xi_0. \tag{18}$$

Using the formula

$$\frac{d}{dx} P_k^{(\alpha,\beta)}(x) = \frac{k+\alpha+\beta+1}{2} P_{k-1}^{(\alpha+1,\beta+1)}(x),$$

which for the ultraspherical orthonormal case yields

$$\frac{d}{dx} \mathbf{P}_k^{(\alpha,\alpha)}(x) = \sqrt{(r-k)k}\; \mathbf{P}_{k-1}^{(\alpha+1,\alpha+1)}(x)$$

and the simplest Taylor expansion around zero,

$$\mathbf{P}_k^{(\alpha,\alpha)}(\xi) = \sqrt{(r-k)k}\; \mathbf{P}_{k-1}^{(\alpha+1,\alpha+1)}(\epsilon\xi)\, \xi, \quad 0 < \epsilon < 1,$$

what reduces the problem to the even case, we obtain

$$F_k^\alpha(\xi) < \sqrt{\delta^2(k,\alpha)-\xi^2}\; (1-\xi^2)^\alpha \left(\mathbf{P}_{k-1}^{(\alpha+1,\alpha+1)}(\epsilon\xi) \right)^2 (r-k)k\, \xi^2 <$$

$$\frac{\sqrt{\delta^2(k,\alpha)-\xi^2}\; (1-\xi^2)^\alpha}{\sqrt{\delta^2(k-1,\alpha+1)-\epsilon^2\xi^2}\; (1-\epsilon^2\xi^2)^{\alpha+1}}\; F_{k-1}^{\alpha+1}(\epsilon\xi)(r-k)k\, \xi_0^2 <$$

$$\frac{\sqrt{\delta^2(k,\alpha)-\xi^2}}{(1-\xi^2)\sqrt{\delta^2(k-1,\alpha+1)-\xi^2}}\; \mathcal{F}_{k-1}^{\alpha+1}\; (r-k)k\, \xi_0^2 \,.$$

The last function increases in ξ and substituting ξ_0 we have

$$F_k^\alpha(\xi) < v(k,\alpha) \mathcal{F}_{k-1}^{\alpha+1}, \tag{19}$$

where

$$v(k,\alpha) = \frac{(r-k)k\, \xi_0^2 \sqrt{\delta^2(k,\alpha)-\xi_0^2}}{(1-\xi_0^2)\sqrt{\delta^2(k-1,\alpha+1)-\xi_0^2}} \,.$$

We have checked using Mathematica that

$$v_1(k,\alpha)=\left(1+\frac{1}{8(k+\alpha)^2}\right)v(k,\alpha)$$

is a decreasing function in k and α, provided $k\ge 3$ and $\alpha\ge\frac12$ (an explicit expression for v is somewhat messy and is omitted). In fact, this is much easier than one may expect as the numerator and the denominator of $\frac{d}{d\alpha}v_1^2(k+3,\alpha+\frac12)$ and $\frac{d}{dk}v_1^2(k+3,\alpha+\frac12)$ consist of the monomials of the same sign.

Calculations yield

$$v_1(3,\frac12)<115\,,\quad v_1(7,\frac{1+\sqrt2}{4})<\frac{29}{2}\,.$$

Finally, applying (14) and (19) and coming back to the usual notation, we conclude

Lemma 10. *Let k be odd, then*

$$\mathcal{M}_k^\alpha(\delta)\le\begin{cases}\frac{230}{\pi} & k\ge 3,\quad \alpha>\frac12\,,\\[2mm] \frac{29}{\pi}\,, & k\ge 7,\quad \alpha>\frac{1+\sqrt2}{4}\,.\end{cases}\tag{20}$$

This completes the proof of Theorem 3.

5. Proof of Theorem 1

First, we will establish the following bounds which are slightly better than these of Theorem 1 but stated in terms of $r=2k+2\alpha+1$, and $\tau=\frac{2\alpha+1}{r}$. It is worth noticing that in some respects r and τ are more natural parameters than k and α (see [7]).

Lemma 11.

$$\mathcal{M}_k^\alpha<\begin{cases}\frac{12}{13}\,r^{1/3}\tan^{1/3}\tau, & k\ge 6,\ \ even,\\[2mm] 14\,r^{1/3}\tan^{1/3}\tau, & k\ge 7,\ \ odd.\end{cases}\tag{21}$$

provided $k\ge 6,\ \ \alpha\ge\frac{1+\sqrt2}{4}$.

Proof. Let $\epsilon=\frac{2^{-1/3}}{3}\,r^{-2/3}\,tan^{4/3}\tau$. It is easy to check that $\epsilon<\frac{1}{31}$, (the extremal case corresponds to $k=6,\ \alpha=\infty$).

Since

$$\delta>\cos\tau>\eta=(1-\epsilon)\cos\tau,$$

where η is defined in (6), it follows by Theorem 2 that all local maxima of $M_k^\alpha(x)$ are inside the interval $(-\delta,\delta)$. Now we have

$$\max_{|x|\le1}\left\{(1-x^2)^{\alpha+\frac12}\left(\mathbf{P}_k^{(\alpha,\alpha)}(x)\right)^2\right\}=\mathcal{M}_k^\alpha(\delta)\max_{0\le x\le\eta}\sqrt{\frac{1-x^2}{\delta^2-x^2}}=\tag{22}$$

$$\mathcal{M}_k^\alpha(\delta)\sqrt{\frac{1-\eta^2}{\delta^2-\eta^2}}\,.$$

By the explicit expression for ϵ given by (6), one can check that the function $\sqrt{2-\epsilon}$ increases in k and a decreases in α. We obtain by $\epsilon<\frac{1}{31}$,

$$\sqrt{\delta^2-\eta^2}>\sqrt{\cos^2\tau-\eta^2}=\sqrt{\epsilon(2-\epsilon)}\,\cos\tau>\frac75\sqrt{\epsilon}\,\cos\tau.$$

Using the restrictions $k \geq 6,\ \alpha \geq \frac{1+\sqrt{2}}{4}$, and a simple trigonometric inequality, we find

$$\sqrt{1-\eta^2} = \sqrt{1-(1-\epsilon)^2\cos^2\tau} \leq \sin\tau\left(1+\epsilon\cot^2\tau\right) =$$
$$\left(1+\frac{1}{3}\left(\frac{2k(k+2\alpha+1)}{(2\alpha+1)^2(2k+2\alpha+1)^2}\right)^{1/3}\right)\sin\tau < \frac{37}{32}\sin\tau.$$

Thus, we obtain

$$\sqrt{\frac{1-\eta^2}{\delta^2-\eta^2}} < \frac{185\tan\tau}{224\sqrt{\epsilon}} = \frac{185\sqrt{3}}{224} r^{1/3}\tan^{1/3}\tau < \frac{13}{9} r^{1/3}\tan^{1/3}\tau,$$

and the result follows by (22) and (14) for k even, and (20) for k odd. □

Now Theorem 1 is an immediate corollary of (21) and

$$\frac{r^{1/3}\tan^{1/3}\tau}{\alpha^{1/3}\left(1+\frac{\alpha}{k}\right)^{1/6}} = \left(\frac{(2\alpha+1)^2(2k+2\alpha+1)^2}{4\alpha^2(k+\alpha)(k+2\alpha+1)}\right)^{1/6} \leq (4\sqrt{2}-2)^{1/3},$$

for $\alpha \geq \frac{1+\sqrt{2}}{4}$. This completes the proof.

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